Embeddings for General Relativity

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Abstract

We present a systematic approach to embed n-dimensional vacuum general relativity in an $(n+1)$-dimensional pseudo-Riemannian spacetime whose source is either a (non)zero cosmological constant or a scalar field minimally-coupled to Einstein gravity. Our approach allows us to generalize a number of results discussed in the literature. We construct *all* the possible (physically distinct) embeddings in Einstein spaces, including the Ricci-flat ones widely discussed in the literature. We examine in detail their generalization, which - in the framework under consideration - are higher-dimensional spacetimes sourced by a scalar field with flat (constant $\neq 0$) potential. We use the Kretschmann curvature scalar to show that many embedding spaces have a physical singularity at some finite value of the extra coordinate. We develop several classes of embeddings that are free of singularities, have distinct non-vanishing self-interacting potentials and are continuously connected (in various limits) to Einstein embeddings. We point out that the induced metric possesses scaling symmetry and, as a consequence, the effective physical parameters (e.g., mass, angular momentum, cosmological constant) can be interpreted as functions of the extra coordinate.

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1 Introduction

Kaluza's great achievement was the discovery that extending the number of dimensions from four to five allows the unification of gravity and electromagnetism [1]. The appearance of the "extra" dimension in physical laws was avoided by imposing the "cylinder condition", which essentially requires that all derivatives with respect to the fifth coordinate vanish.

Modern higher-dimensional gravity theories, inspired by string theories, do not require this condition; it is replaced by the assumption that matter fields are confined to our 4D spacetime, which is modeled as a singular hypersurface or "brane" embedded in a larger $(4 + d)$ world, while gravity is a multidimensional interaction that can propagate in the extra d dimensions as well [2–4]. Well-known examples of such brane theories are provided by the Randall-Sundrum [5, 6] and the Dvali-Gabadadze-Porrati scenarios [7, 8]. In the so-called space-time-matter theory, inspired by the unification of matter and geometry, the framework is similar; the fifth dimension is not compactified and our spacetime is identified with some four-dimensional hypersurface orthogonal to the extra dimension. The main difference is that in the latter the hypersurface is not necessarily singular and the effective matter in 4D is derived from vacuum in 5D, through the metric's dependence on the extra coordinate [9–12].

Independently of the theoretical motivation, the existence of a putative large extra dimension offers a wealth of new physics. That is illustrated by the effective equations for gravity in 4D, which predict five-dimensional local and non-local corrections to the usual general relativity in 4D [13–22]. Also, the geodesic equation for test particles in 5D predicts an effective four-dimensional equation of motion with an extra non-gravitational force [23–30]. As a consequence, there has been a growing interest in models where our 4D spacetime is embedded in a higher-dimensional space (for historical reviews and references see [31, 32]), which in turn has generated considerable attention to embedding theorems of differential geometry and their applications to higher-dimensional theories of the universe.

In particular, Campbell-Magaard's theorem [33], which asserts that any n-dimensional pseudo-Riemannian manifold may be locally embedded in an $(n+1)$- dimensional Ricci-flat space, implies that it is always possible to locally embed any solution of the 4D Einstein equations of general relativity - with an arbitrary energy-momentum tensor - in a Ricci-flat solution of the 5D vacuum Einstein equations. Motivated by the second Randall-Sundrum braneworld scenario and string theories, the mathematical proof of that theorem has been extended to include cases where the higher-dimensional space has a nonzero cosmological constant, is sourced by a scalar field, and has an arbitrary non-degenerate Ricci tensor [34–40]. However, it is important to note two things: First, embedding theorems provide the conditions that guarantee the existence of the embedding, but they do not show *how* to produce the actual embedding. Second, there are many ways of embedding a 4D spacetime in a higher-dimensional manifold [41, 42].

A neat example of this is provided by the problem of embedding vacuum solutions of n-dimensional general relativity in an $(n+1)$-dimensional manifold. In fact, there are a number of works proposing specific embeddings of such solutions in Ricci-flat spaces, Einstein spaces, and spaces sourced by a massless minimally-coupled scalar field. A natural question to ask is whether these (seemingly disconnected) embeddings are related to each other, and if so how? In this paper we present a systematic approach that allows us to tackle all these embeddings in a unified manner in such a way that we (i) reproduce known results, (ii) construct new embeddings and (iii) give concrete answers to the above-mentioned questions. Our results complement and generalize those obtained in [36, 43–48].

The paper is organized as follows. In Section 2 we provide the field equations and perform the dimensional reduction. We show that there are infinite ways of building these embeddings as there are infinite ways of prescribing the embedding function. In Section 3 we construct *all* the possible embeddings in Einstein spaces, including the Ricci-flat ones discussed in [43,44]. By construction they are free of the degeneracy noted by Fonseca-Neto and Romero [46]. In Section 4 we examine in detail the simplest generalization of Einstein embeddings, which - in the framework under consideration - are spaces with one extra dimension sourced by a scalar field with flat (constant $\neq 0$) potential. We use the Kretschmann curvature scalar, which we calculate in Appendix A, to show that many embedding spaces have a physical singularity at some finite value of the extra coordinate. In Section 5 we develop several classes of embeddings that are free of singularities, have distinct non-vanishing self-interacting potentials and are continuously connected (in various limits) to the Einstein embeddings of Section 3. In Section 6 we summarize some of the physical implications. We point out that the induced metric possesses scaling symmetry and, as a consequence, the effective physical parameters (e.g., mass, angular momentum, cosmological constant) can be interpreted as functions of the extra coordinate. We show that the non-vanishing logarithmic derivative of the embedding function is responsible

for the appearance of an extra force in n-dimensions.

2 Field equations and dimensional reduction

Let us consider an n-dimensional Lorentzian manifold $(\Sigma,\, g_{ab})$, which is a solution of the n-dimensional Einstein equations of general relativity in vacuum with cosmological constant $\Lambda_{(n)}$, i.e.,

$$^{(n)}G_{ab} = \Lambda_{(n)}\, g_{ab}, \quad (a,\, b = 0,\, 1,\, \dots,\, n-1). \tag{1}$$

Here, we will show *how* such a manifold may be embedded in a solution of the Einstein equations in $(n+1)$ dimensions

$$^{(n+1)}G_{AB} = \kappa\, ^{(n+1)}T_{AB}, \quad (A,\, B = 0,\, 1,\, \dots,\, n), \tag{2}$$

where κ represents the gravitational constant in $(n+1)$ dimensions and $^{(n+1)}T_{AB}$, following Anderson *et al.* [45], is the energy-momentum of a real massless scalar field Ψ, minimally-coupled to Einstein gravity and self-interacting through a potential $V(\Psi)$. The energy momentum tensor of such a field is

$$^{(n+1)}T_{AB} = \Psi_A\, \Psi_B - \frac{1}{2}\gamma_{AB}\, \Psi_C \Psi^C - \gamma_{AB}\, V(\Psi). \tag{3}$$

Here Ψ is dimensionless, V has the units of $(\text{length})^{-2}$ and γ_{AB} is the metric of the $(n+1)$-dimensional space. In this framework we will be able to expand and generalize a number of previous results found in the literature.

To construct an embedding for $(\Sigma,\, g_{ab})$ it is convenient to work in Gaussian normal coordinates where the line element in $(n+1)$ has the form

$$dS^2 = \gamma_{AB}\, dx^A dx^B \quad = \quad \gamma_{ab}\, dx^a dx^b + \epsilon dy^2, \tag{4}$$

where y represents the coordinate along the $(n+1)$-th dimension and $\epsilon = -1$ or $\epsilon = 1$ depending on whether it is spacelike or timelike, respectively. Next, we adopt the ansatz used by Mashhoon and Wesson [43, 44] and split γ_{ab} into two parts via

$$\gamma_{ab} = \mathrm{e}^{2\beta(x^\rho, y)}\, g_{ab}(x^\rho). \tag{5}$$

Now we proceed to solve the field equations (2)-(3) subject to this split.

To compute the components of the Ricci tensor, we first calculate the Christoffel symbols. From (4)-(5) it follows that $\gamma^{ab} = g^{ab}\, \mathrm{e}^{-2\beta}$, $\gamma^{yy} = \gamma_{yy} = \epsilon$ and $\gamma^{ya} = 0$. Thus, we obtain

$$\begin{aligned} K^y_{yy} &= K^a_{yy} = K^y_{ay} = 0 \\ K^b_{ya} &= \beta'\, \delta^b_a, \quad K^y_{ab} = -\epsilon\, \beta'\, \mathrm{e}^{2\beta}\, g_{ab} \\ K^c_{ab} &= \Gamma^c_{ab} + \left(\beta_a\, \delta^c_b + \beta_b\, \delta^c_a - \beta^c\, g_{ab}\right), \end{aligned} \tag{6}$$

where K^C_{AB} and Γ^c_{ab} are the Christoffel symbols formed from γ_{AB} and g_{ab}, respectively; $\beta' = (\partial\beta/\partial y)$; $\beta_a = (\partial\beta/\partial x^a)$ and $\beta^a = g^{ab}\,(\partial\beta/\partial x^b)$.

Consequently, the Ricci tensor for the metric (4)-(5) can be written as

$$^{(n+1)}R_{ya} = -(n-1)\, \beta'_a \tag{7}$$

$$^{(n+1)}R_{yy} = -n\left(\beta'' + \beta'^2\right) \tag{8}$$

$$^{(n+1)}R_{ab} = {}^{(n)}R_{ab} + \sigma_{ab} - \epsilon\left(\beta'' + n\beta'^2\right)\mathrm{e}^{2\beta}\, g_{ab}, \tag{9}$$

where

$$\sigma_{ab} = (2-n)\, \left[\beta_{a;b} - \beta_a\, \beta_b\right] - g_{ab}\, \left[(n-2)\, \beta_c\, \beta^c + \beta^c_c\right]. \tag{10}$$

Here $\beta_{a;b}$ is the covariant derivative constructed with Γ^c_{ab} and $\beta^c_c = g^{ab}\,\beta_{a;b}$.

The field equations (2)-(3) imply

$$^{(n+1)}R_{AB} = \kappa\,\left[\Psi_A\,\Psi_B + \frac{2V}{(n-1)}\,\gamma_{AB}\right]. \tag{11}$$

In what follows $n \neq 1$, because there is no Einstein gravity in one dimension (the concept of curvature requires at least two dimensions). Since $\gamma_{ya} = 0$, from this and (7) it follows that $(n-1)\,\beta'_a = -\kappa\,\Psi_a\,\Psi'$. This equation admits immediate integration if $\Psi =$ constant, or in a more general case when either $\Psi' = 0$ or $\Psi_a = 0$. To make contact with [43–45], we assume here that $\Psi_a = 0$, which means that Ψ is either a constant or only depends on y. As a consequence $^{(n+1)}R_{ay} = 0$ and we obtain

$$\mathrm{e}^{2\beta(x^\rho, y)} = F(y)\,\mathrm{e}^{2f(x^\rho)}, \tag{12}$$

where $F(y)$ and $f(x^\rho)$ are arbitrary functions of integration. In addition from (9) and (11) we get

$$^{(n)}R_{ab} = \left\{\frac{\epsilon}{2}\,\left[F'' + \frac{(n-2)}{2}\,\frac{F'^2}{F}\right] + \frac{2\,\kappa\,V}{(n-1)}\,F\right\}\,\mathrm{e}^{2f}\,g_{ab} - \sigma_{ab}. \tag{13}$$

If $f = f_0 =$ constant, then $\sigma_{ab} = 0$ and the embedded metric g_{ab} represents an Einstein space. In this case, from (1) it follows that the term multiplying the metric is $[2\Lambda_{(n)}/(2-n)]$. Thus we find

$$\kappa\,V = -\frac{\epsilon\,(n-1)}{4}\,\left[\frac{F''}{F} + \frac{(n-2)}{2}\,\frac{F'^2}{F^2} + \frac{4\,\epsilon\tilde{\Lambda}_{(n)}}{(n-2)F}\right], \tag{14}$$

where $\tilde{\Lambda}_{(n)} = \Lambda_n\,\mathrm{e}^{-2f_0}$. In what follows we omit the tilde.

Similarly, equating the expressions for $^{(n+1)}R_{yy}$ from (8) and (11), and using (14), we find

$$\kappa\Psi'^2 = -\frac{(n-1)}{2}\,\left[\frac{F''}{F} - \frac{F'^2}{F^2}\right] + \frac{2\,\epsilon\Lambda_{(n)}}{(n-2)\,F}. \tag{15}$$

Thus, the potential and the scalar field are completely determined by the function $F(y)$.

Without loss of generality we can set $f_0 = 0$, which is equivalent to a change of scale of coordinates, viz., $\mathrm{e}^{f_0}\,x \to x$. After such a change, the line element in $(n+1)$ becomes

$$dS^2 = F(y)\,g_{ab}\,dx^a dx^b + \epsilon dy^2, \tag{16}$$

which for every given F provides an embedding for $(\Sigma,\, g_{ab})$. The field equations do not give an equation for F, which reflects the fact that there are an infinite ways of embedding an n-dimensional manifold in an $(n+1)$-dimensional space. To investigate the presence of singularities in these embeddings, in Appendix A we evaluate the Kretschmann scalar for metric (16).

To complete the discussion we should mention that (14) and (15) are consistent with the equation of motion of the scalar field

$$\Psi^{C}_{;C} - \frac{dV}{d\Psi} = 0. \tag{17}$$

In fact, for the line element (16) this equation reduces to

$$\Psi'\Psi'' + \frac{n}{2}\,\frac{F'}{F}\,\Psi'^2 - \epsilon V' = 0, \tag{18}$$

which is identically satisfied by (14) and (15). This is what we expected because the equations of the gravitational field contain the equations for the matter which produces the field.

Also, we provide the non-vanishing mixed components of the energy-momentum tensor (3). They are

$$\kappa\ {}^{(n+1)}T^0_0 = \frac{\epsilon\,(n-1)}{2}\left[\frac{F''}{F} + \frac{(n-4)}{4}\frac{F'^2}{F^2}\right] + \frac{\Lambda_{(n)}}{F} \tag{19}$$

$$\kappa\ {}^{(n+1)}T^y_y = \frac{\epsilon\, n(n-1)}{8}\frac{F'^2}{F^2} + \frac{n\,\Lambda_{(n)}}{(n-2)\,F}, \tag{20}$$

and $T^0_0 = T^1_1 = \cdots = T^n_n$ (no summation). In what follows we set $\kappa = 1$.

In the above equations $n \neq 2$. However, this does not impose physical restrictions because in two-dimensions the Einstein tensor vanishes identically, so the cosmological constant $\Lambda_{(2)}$ must be zero. Therefore, if one wants to use these equations for $n = 2$ one should set $\left[\Lambda_{(n)}/(n-2)\right]_{n=2} = 0$ everywhere.

2.1 Checking the equations

To check the above equations we apply them to the case where $F(y) = (1+\lambda y)^{2p}$ and $\Lambda_{(n)} = 0$ considered by Anderson *et al.* [45]. Here p and λ are constants. Substituting into (15) we get

$$\Psi'^2 = \frac{p\,(n-1)\lambda^2}{(1+\lambda y)^2}. \tag{21}$$

Thus,

$$\Psi = q\,\ln|1+\lambda y| + \Psi_0, \tag{22}$$

where Ψ_0 is a constant of integration and

$$q^2 = p\,(n-1) \tag{23}$$

The corresponding potential (14) is

$$V(\Psi) = -\frac{\epsilon\, q^2\,\lambda^2\,(np-1)}{2}\,\mathrm{e}^{-2(\Psi-\Psi_0)/q}. \tag{24}$$

Expressions (22)-(24) reproduce the results of section 3.2 of [45] (in that paper $\epsilon = 1$).

3 Einstein embeddings: Ψ = constant

The simplest way to construct a particular embedding for the metric g_{ab} is to prescribe the function $F(y)$. However, we have no physical arguments which would allow us to decide in favor of some particular choice.

In this section we assume Ψ = constant, which means that the embedding universe is an Einstein space with, in general, nonzero Ricci curvature. This assumption provides an equation for F. Namely, from (15) we get

$$FF'' - F'^2 - \frac{4\,\epsilon\,\Lambda_{(n)}}{(n-1)(n-2)}\,F = 0. \tag{25}$$

The first integral to this equation is

$$F'^2 = C\,F^2 - \frac{8\,\epsilon\,\Lambda_{(n)}}{(n-1)(n-2)}\,F, \tag{26}$$

where C is a constant of integration with dimensions of (length)$^{-2}$. Substituting this into (14) we find

$$V = -\frac{\epsilon\,C\,n(n-1)}{8}. \tag{27}$$

In the case under consideration the energy-momentum tensor (3) reduces to ${}^{(n+1)}T_{AB} = -V\gamma_{AB}$, which implies that $-V$ can be interpreted as the cosmological constant $\Lambda_{(n+1)}$ in $(n+1)$-dimensions. Thus,

$$C = \frac{8\epsilon\,\Lambda_{(n+1)}}{n(n-1)}. \tag{28}$$

Consequently, when $\Psi =$ constant the function F is determined by the equation

$$F'^2 = \frac{8\,\epsilon\,F}{n-1}\left[\frac{\Lambda_{(n+1)}}{n}\,F - \frac{\Lambda_{(n)}}{n-2}\right]. \tag{29}$$

3.1 The function $F(y)$ for Einstein embeddings

Equation (29), for $n \neq 2$, has seven distinct types of solutions that depend on $\Lambda_{(n)}$, $\Lambda_{(n+1)}$ and ϵ. For $n = 2$ it has a unique solution which is given by (30) below.

Type I: The simplest solution is $F = F_0 =$ constant. From (25) and (29) it follows that F is constant only if $\Lambda_{(n)} = \Lambda_{(n+1)} = 0$.

Type II: These are solutions for $\Lambda_{(n)} = 0$, $\Lambda_{(n+1)} \neq 0$. From (29) we find that this case requires $\epsilon\,\Lambda_{(n+1)} > 0$, which means that the $(n+1)$-dimensional spacetime is anti-de Sitter (de Sitter) if the extra dimension is spacelike (timelike). The solution is

$$F(y) = F_0\,\mathrm{e}^{s\sqrt{C}\,y}, \quad C > 0, \quad s = \pm 1, \tag{30}$$

where F_0 is a constant of integration and C is given by (28). When $\Lambda_{(n+1)} \to 0$ we recover the constant solution.

This type of embedding is reminiscent of the reduction ansatz used in the RS-2 scenario of braneworld theory[1] [6]. It allows to embed any solution of the n-dimensional Einstein field equations in vacuum with zero cosmological constant in an $(n+1)$-dimensional Einstein space with a nonzero cosmological constant.

Type III: These are solutions for $\Lambda_{(n+1)} = 0$, $\Lambda_{(n)} \neq 0$. In this case the $(n+1)$-spacetime is usually called Ricci-flat. To obtain solutions to (29) in terms of real functions we should take $\epsilon\Lambda_{(n)} < 0$. Also, to simplify the notation we introduce the quantity

$$\bar{\Lambda}_{(n)} = \frac{6\Lambda_{(n)}}{(n-1)(n-2)}, \tag{31}$$

in terms of which the solution can be written as

$$F(y) = \left[\sqrt{\frac{-\epsilon\,\bar{\Lambda}_{(n)}}{3}}\,y + \sqrt{F_0}\right]^2. \tag{32}$$

It reduces to the constant solution when $\Lambda_{(n)} \to 0$. This embedding generalizes to an arbitrary number of dimensions the ones obtained in [43, 44, 46].

When $\Lambda_{(n)} \neq 0$ and $\Lambda_{(n+1)} \neq 0$ the solutions of (29) crucially depend on the sign of $\epsilon\,\Lambda_{(n+1)}$. If $\epsilon\,\Lambda_{(n+1)} > 0$, then $\epsilon\,\Lambda_{(n)}$ can be positive, negative or zero and the solutions are expressed in terms of exponentials. If $\epsilon\,\Lambda_{(n+1)} < 0$, then $\epsilon\bar{\Lambda}_{(n)}$ must be negative and the solutions are oscillatory. In addition, their behavior depends on the choice of the remaining constant of integration. Here we choose it in such a way that we recover the solutions of Type I - Type III in the appropriate limits.

[1] In that scenario y in (30) is replaced by its modulus, so that F' is discontinuous at $y = 0$, which gives a delta-function contribution to ${}^{(n+1)}R_{AB}$.

Type IV: These are solutions for $\epsilon\,\Lambda_{(n+1)} > 0$ and $\Lambda_{(n)} \neq 0$, which contain those of Type II as a limiting case ($s\sqrt{C}\,y \gg 1$). There are three "subtypes".

• **Type IVa**. These are embeddings that admit both signs of $\epsilon\,\bar{\Lambda}_{(n)}$. In the limit $\Lambda_{(n)} \to 0$ they yield (30), independently of $\Lambda_{(n+1)}$ (or C). Namely,

$$F(y) = F_0\,\mathrm{e}^{s\sqrt{C}\,y}\left[1 + \frac{\epsilon\,\bar{\Lambda}_{(n)}}{3F_0\,C}\,\mathrm{e}^{-s\sqrt{C}\,y}\right]^2, \qquad C \neq 0. \tag{33}$$

• **Type IVb**. These embeddings require $\epsilon\,\bar{\Lambda}_{(n)} < 0$, and in the limit $\Lambda_{(n+1)} \to 0$ reproduce (32), viz.,

$$F(y) = -\frac{4\,\epsilon\,\bar{\Lambda}_{(n)}}{3C}\,\sinh^2\left[\frac{\sqrt{C}\,(y+\tilde{y})}{2}\right], \quad \epsilon\,\bar{\Lambda}_{(n)} < 0, \tag{34}$$

where $\tilde{y}$ is a constant. If we choose $\tilde{y} = \left(-3\,\epsilon\,F_0/\bar{\Lambda}_n\right)^{1/2}$, then in the limit $C \to 0$ ($\Lambda_{(n+1)} \to 0$) we recover the Ricci-flat embedding (32). When $\epsilon = -1$ this function allows to embed any solution of the n-dimensional Einstein field equations in vacuum with $\Lambda_{(n)} > 0$ in an $(n+1)$-dimensional Einstein space with $\Lambda_{(n+1)} < 0$.

• **Type IVc**. These embeddings require $\epsilon\,\bar{\Lambda}_{(n)} > 0$ and do not support the limits $\Lambda_{(n+1)} \to 0$, $\Lambda_{(n)} \to 0$, separately. Namely,

$$F(y) = \frac{4\,\epsilon\,\bar{\Lambda}_{(n)}}{3C}\,\cosh^2\left[\frac{\sqrt{C}\,(y+\tilde{y})}{2}\right], \quad \epsilon\bar{\Lambda}_{(n)} > 0. \tag{35}$$

However, if we assume that the cosmological constants are related, e.g., as

$$\Lambda_{(n)} = F_0\,\frac{(n-2)}{n}\,\Lambda_{(n+1)}, \qquad F_0 = \text{constant} > 0 \tag{36}$$

then (35) becomes (without loss of generality we set $\tilde{y} = 0$)

$$F(y) = F_0\,\cosh^2\left(\frac{\sqrt{C}\,y}{2}\right), \quad \epsilon\bar{\Lambda}_{(n)} > 0. \tag{37}$$

which in the limit $\Lambda_{(n)} \to 0$ ($\Lambda_{(n+1)} \to 0$) is well-behaved and reproduces the constant (Type I) solution. When $\epsilon = -1$ this function allows to embed any solution of the n-dimensional Einstein field equations in vacuum with $\Lambda_{(n)} < 0$ in an $(n+1)$-dimensional Einstein space with $\Lambda_{(n+1)} < 0$. By virtue of the relation (36), here $\Lambda_{(n)}$ is an adjustable parameter; by selecting F_0 appropriately it can be chosen to be small enough to be phenomenologically realistic, regardless of the size of $\Lambda_{(n+1)}$ [49].

Type V: These are the trigonometric counterparts of (34), (35) and (37), which are the solutions to (29) when $\epsilon\,\Lambda_{(n+1)} < 0$. All of them require $\epsilon\bar{\Lambda}_{(n)} < 0$, since F is nonnegative.

The counterpart of (34) is a new solution, viz.,

$$F(y) = -\frac{4\,\epsilon\,\bar{\Lambda}_{(n)}}{3\omega^2}\,\sin^2\left[\frac{\omega\,(y+\tilde{y})}{2}\right], \quad \omega = \sqrt{-C} > 0, \quad \epsilon\,\bar{\Lambda}_{(n)} < 0. \tag{38}$$

In the limit $C \to 0$ ($\Lambda_{(n+1)} \to 0$) it reduces to the Ricci-flat embedding (32). However, the trigonometric counterpart of (35), with $C = -\omega^2$, is not a new solution because after a simple change $\tilde{y} \to (\tilde{y} - \pi/\omega)$ it reduces to (38). If we apply the same procedure to (37) we find that

$$F(y) = F_0\,\cos^2\left(\frac{\omega\,y}{2}\right), \quad \omega = \sqrt{-C}, \tag{39}$$

is an embedding function only if (36) holds true, viz., $\omega = \sqrt{-C} = \sqrt{-\frac{4\,\epsilon\bar{\Lambda}_{(n)}}{3F_0}}$. For $\epsilon = -1$, (38) and (39) serve to embed n-dimensional vacuum solutions with $\Lambda_{(n)} > 0$ in an $(n+1)$-dimensional Einstein space with $\Lambda_{(n+1)} > 0$. The difference is that $\Lambda_{(n)}$ and $\Lambda_{(n+1)}$ are independent parameters in (38), but not in (39).

3.2 Singularities of the Einstein embeddings

To investigate the singularities we use the Kretschmann curvature scalar. Substituting (25)-(26) into (A-8) we obtain

$$\widehat{K}^2 = \frac{1}{F^2}\left[K^2 - \frac{2}{9}\,n(n-1)\,\bar{\Lambda}^2_{(n)}\right] + \frac{8\,(n+1)}{n(n-1)^2}\,\Lambda^2_{(n+1)}. \tag{40}$$

Note that for embeddings of Type II, F becomes zero only asymptotically, as $(s\,y) \to -\infty$. Those of Type IV with $\epsilon\Lambda_{(n)} > 0$ have $F(y) > 0$, for all values of y. However, F does reach zero at some finite value of y, say $y = y_*$, when $\epsilon\Lambda_{(n)} < 0$. In the latter case, $y = y_*$ is a curvature singularity, unless the term in square bracket is zero. As far as we know, the only case where this is so is the n-dimensional de Sitter space.[2] So the embeddings of[3] dS_n or AdS_n in dS_{n+1} or AdS_{n+1} are the only ones which are free of singularities at y_*; the embedding of any other metric, for example the Schwarzschild-de Sitter metric, will be singular at those hypersurfaces.

It may be verified that the behavior of all solutions with $\epsilon\,\Lambda_{(n)} < 0$, namely (32)-(34) and (38)-(39), near y_* is

$$F(y) \overset{y\to y_*}{\longrightarrow} -\frac{\epsilon\,\bar{\Lambda}_{(n)}}{3}\,(y-y_*)^2. \tag{41}$$

With the transformation $\tilde{y} = (y - y_*)$, which is a change to a new zero point for y, the line element (16) near y_* becomes (omitting the tilde)

$$dS^2 \overset{y\to 0}{\longrightarrow} -\frac{\epsilon\,\bar{\Lambda}_{(n)}}{3}\,y^2\,g_{ab}\,dx^a\,dx^b + \epsilon\,dy^2, \qquad \epsilon\,\bar{\Lambda}_{(n)} < 0. \tag{42}$$

When $n = 4$ and $\epsilon = -1$ this reduces to the (pure) canonical metric employed by Mashhoon and Wesson [43, 44] to embed vacuum solutions with $\Lambda_{(4)} > 0$ in a five-dimensional Ricci-flat space. Consequently, we can assert that the (pure) canonical metric in $(n+1)$-dimensions (42) describes the asymptotic behavior of Einstein embeddings, with $\Lambda_{(n+1)} \neq 0$, near the hypersurface $y = 0$, which - except for dS_n and AdS_n - is a singular one for all embeddings of vacuum solutions of n-dimensional general relativity.

4 Flat potential: $V(\Psi) = V_0 =$ constant

This is the simplest generalization of the Einstein embeddings discussed in the previous section. In fact, when $V = V_0 =$ constant, (14) provides and equation for F

$$FF'' + \frac{(n-2)}{2}\,F'^2 + \frac{4\epsilon\Lambda_{(n)}}{(n-2)}\,F + \frac{4\,\epsilon V_0\,F^2}{(n-1)} = 0, \tag{43}$$

whose first integral is

$$F'^2 = \frac{E}{F^{(n-2)}} - \frac{8\epsilon\,\Lambda_{(n)}\,F}{(n-1)(n-2)} - \frac{8\epsilon\,V_0\,F^2}{n(n-1)}, \tag{44}$$

where E is a constant of integration with dimensions of $(\text{length})^{-2}$. Inserting this into (15) we find

$$\Psi'^2 = \frac{E\,n(n-1)}{4F^n}, \qquad E \geq 0, \tag{45}$$

which is consistent with (18). Note that $\Psi =$ constant when $E = 0$, and (44) becomes identical to (29) with a cosmological constant in the $(n+1)$-dimensional space given by $\Lambda_{(n+1)} = -V_0$, as expected. Thus, the Einstein embeddings correspond to $E = 0$.

[2] In fact, the Riemann curvature tensor of de Sitter space in n-dimensions is given by ${}^{(n)}R_{abcd} = \frac{2\,\Lambda_{(n)}}{(n-1)(n-2)}\,(g_{ad}\,g_{bc} - g_{ac}\,g_{bd})$. In terms of $\bar{\Lambda}_{(n)}$, introduced in (31), we obtain $K^2 = \frac{2}{9}\,n(n-1)\,\bar{\Lambda}^2_{(n)}$.

[3] $dS_n \equiv$ de Sitter spacetime in n-dimensions; $AdS_n \equiv$ Anti-de Sitter spacetime in n-dimensions.

Unfortunately there is no a general algebraic solution to these equations, but we can construct a mechanical analog which helps us to understand the nature of the solutions. To this end we set

$$F = X^{2/n}, \tag{46}$$

which allows to write (44) as

$$\frac{1}{2}\,\mu\, X'^2 = E - U(X), \tag{47}$$

where $\mu = (8/n^2)$ and

$$U(X) = \frac{8\,\epsilon}{n-1}\left[\frac{\Lambda_{(n)}}{(n-2)}\,X^{2(n-1)/n} + \frac{V_0}{n}\,X^2\right], \tag{48}$$

which in the language of classical mechanics describes a particle of mass μ moving in the X-direction, under the action of a X-directed conservative force with potential $U(X)$, and total constant energy E.

4.1 The embedding function $F(y)$ for flat potential

Once again, there are seven distinct types of solutions depending on $\epsilon\, V_0$ and $\epsilon\, \Lambda_{(n)}$.

1) Solutions for $\Lambda_{(n)} \neq 0$ and $F = F_0$ constant: This is the simplest solution to (43)-(45). It is given by

$$\Psi(y) = \sqrt{\frac{2\epsilon\,\Lambda_{(n)}}{(n-2)\,F_0}}\; y + \Psi_0, \qquad \epsilon\,\Lambda_{(n)} > 0 \tag{49}$$

$$V_0 = -\frac{(n-1)\,\Lambda_{(n)}}{(n-2)\,F_0}, \qquad \frac{\Lambda_{(n)}}{V_0} < 0 \tag{50}$$

$$E = \frac{4\,\epsilon\,\bar{\Lambda}_{(n)}}{3\,n}\left[\left(\frac{n-1}{n-2}\right)\left(-\frac{\Lambda_{(n)}}{V_0}\right)\right]^{(n-1)}, \tag{51}$$

where to simplify the notation we have used (31). They describe the unstable equilibrium position at the top of the curve $U(X)$, see solutions 7 below. In the limit $\Lambda_{(n)} \to 0$ we recover the solutions of Type I.

2) Solutions for $\Lambda_{(n)} = 0$ and $V_0 = 0$: In this case $U = 0$, which describes a mechanical system in inertial motion. The solution in terms of the field quantities is

$$F(y) = \left|\frac{n\,\sqrt{E}}{2}\,y + 1\right|^{2/n} \tag{52}$$

$$\Psi(y) = \pm\sqrt{\frac{n-1}{n}}\,\ln\left|\frac{n\,\sqrt{E}}{2}\,y + 1\right| + \Psi_0. \tag{53}$$

For $\frac{n\sqrt{E}}{2} = \lambda$ the solution is identical to that of Anderson *et al.* [45], given by (22), with $p = 1/n$ for which the potential (24) vanishes. Changing the origin $y \to (y - 2/n\sqrt{E})$ it can be written as

$$F(y) = \left(\frac{n\,\sqrt{E}}{2}\,y\right)^{2/n}, \quad \Psi(y) = \pm\sqrt{\frac{n-1}{n}}\,\ln\left(\frac{n\,\sqrt{E}}{2}\,|y|\right). \tag{54}$$

3) Solutions for $\Lambda_{(n)} = 0$ and $\epsilon\, V_0 > 0$: Now U is positive and quadratic in X. Therefore, (47)-(48) represent a simple harmonic oscillator. The corresponding solution for F and Ψ is given by

$$F(y) = \left[\frac{n\sqrt{E}}{2\,\omega}\,\sin(\omega\, y + \phi)\right]^{2/n}, \quad \omega = \sqrt{\frac{2\,n\,\epsilon\, V_0}{n-1}} \tag{55}$$

$$\Psi(y) = \sqrt{\frac{n-1}{n}}\,\ln\tan\left(\frac{\omega\, y + \phi}{2}\right) + \Psi_0, \tag{56}$$

where ϕ is a constant of integration. Here the extra dimension y is restricted to the range $0 < (\omega\, y + \phi) < \pi$.

4) Solutions for $\Lambda_{(n)} = 0$ and $\epsilon\, V_0 < 0$: Again U is quadratic in X, but this time it is negative. Therefore the functions F and Ψ have the same mathematical shape as in (55)-(56), except that now the trigonometric functions are replaced by the corresponding hyperbolic functions and $\omega = \sqrt{-2\,n\epsilon\, V_0/(n-1)}$. The physical consequence is that the extra dimension y can take arbitrary large values. In such a case, $F(y) \to \mathrm{e}^{2\omega y/n}$ and $\Psi' \to 0$ which implies ${}^{(n+1)}T_{AB} \to -\gamma_{AB}\, V_0 = \gamma_{AB}\, \Lambda_{(n+1)}$. Then, by virtue of the relation between ω and V_0, for large values of $(\omega\, y + \phi)$ these solutions approach those of Type II discussed in the previous section.

- If we set $\phi = (2\,\omega/n\,\sqrt{E})$, then in the limit $V_0 \to 0$ the last two solutions exactly reproduce (52)-(53).

5) Solutions for $V_0 = 0$ and $\epsilon\, \Lambda_{(n)} < 0$: In this case $U(X) < 0$ and the motion is unbounded, i.e., $0 < X < \infty$. The solutions cannot be expressed in terms of elementary functions, except for $E = 0$. But, their qualitative behavior is as follows: Near the origin they are well represented by (54). For $X \gg 1$ they mimic those of Type III (32) for which $\Lambda_{(n+1)} = 0$.

6) Solutions for $V_0 = 0$ and $\epsilon\, \Lambda_{(n)} > 0$: Now $U(X) > 0$, the (fictitious) particle moves under the action of a nonlinear restoring force $\sim -X^{(n-2)/n}$, and X is bounded to move between zero and X_{max}, which is given by the solution of the equation $E = U(X)$. In terms of F we have $0 \leq F \leq \left(\frac{3E}{4\,\epsilon\Lambda_{(n)}}\right)^{1/(n-1)}$.

7) Solutions for $V_0 \neq 0$ and $\Lambda_{(n)} \neq 0$: We introduce the dimensionless parameter α as

$$\frac{V_0}{n} = \alpha\,\frac{\Lambda_{(n)}}{(n-2)}. \tag{57}$$

Then, using the quantity $\bar{\Lambda}_{(n)}$ introduced in (31), the function U becomes

$$U(X) = \frac{4}{3}\,\epsilon\,\bar{\Lambda}_{(n)}\,\left(X^{2(n-1)/n} + \alpha X^2\right).$$

There are different solutions depending on the sign of $\epsilon\,\Lambda_{(n)}$ and α.

Perhaps the most interesting are those with $\epsilon\,\Lambda_{(n)} > 0$ and $\alpha < 0$. In that case U vanishes not only at $X = 0$ but also at $X = (-1/\alpha)^{n/2}$. Thus, U has a maximum $U_0 = \frac{4\,\epsilon\,\bar{\Lambda}_{(n)}}{3\,n}\left[\frac{(n-1)}{(-\alpha)\,n}\right]^{(n-1)}$ at $X_0 = \left[\frac{n-1}{(-\alpha)\,n}\right]^{n/2}$. Consequently, for $E < U_0$ the motion is bounded. For $E > U_0$, X (or F) can take arbitrary large values. Asymptotically, for $F \gg 1$ the solutions coincide with those of Type II. For $E = U_0$, the system eventually reaches the equilibrium point X_0 where $X' = X'' = 0$, which corresponds to a state of unstable equilibrium, since $U(X)$ has a maximum at that point. Such a state is described by (49)-(51).

For $\epsilon\,\Lambda_{(n)} < 0$ and $\alpha < 0$ the solutions are oscillatory and generalize those of Type V to the case where $E > 0$.

4.2 Singularities of the embeddings with flat potential

To obtain the Kretschmann scalar for these embeddings we substitute (43)-(44) into (A-8) and get

$$\widehat{K}^2 = \frac{1}{F^2}\left[K^2 - \frac{2}{9}\,n(n-1)\,\bar{\Lambda}^2_{(n)}\right] + \frac{8(n+1)}{n(n-1)^2}\,V_0^2 + \frac{2\,E}{F^n}\left[\epsilon\,V_0 + \frac{E\,(2n-1)(n-1)n}{16\,F^n}\right]. \tag{58}$$

For $E = 0$ we recover (40), as expected. Except for (49)-(51), all solutions of (43)-(44) with $E > 0$ become zero at some finite $y = y_*$, which is a curvature singularity even when the embedded spacetime is dS_n or AdS_n, and the first square bracket vanishes. We can choose the origin of y to make $F(0) = 0$. With this choice, the behavior of the solutions near the singularity is given by (54) and the line element is

$$dS^2 \overset{y\to 0}{\longrightarrow} \left(\frac{n\,\sqrt{E}}{2}\,y\right)^{2/n} g_{ab}\,dx^a\,dx^b + \epsilon\,dy^2, \tag{59}$$

regardless of $\Lambda_{(n)}$ and V_0.

5 General case: $\Psi \neq$ constant, $V \neq$ constant

As mentioned earlier, we have a system of two equations, namely (14) and (15), for three unknown functions, $(F,\,\Psi,\,V)$. One way to close the system is to prescribe a function $V = V(\Psi)$. However, in absence of such a function, a more appealing procedure is to assume either $\Psi =$ constant or $V =$ constant, which is what we have done in the last two Sections. Otherwise, there are infinite ways of prescribing the embedding function F.

In this section, we build several examples. To reduce the arbitrariness we select F in such a way that, in the appropriate limits, we regain some of the previously discussed embedding spacetimes.

Example I: As a model for the choice of F we use (30). Thus we assume

$$F = F_0\,\mathrm{e}^{\lambda\,y}, \tag{60}$$

where λ is a parameter with dimensions of $(\text{length})^{-1}$. In this case $F \neq 0$ for any finite value of y and the Kretschmann scalar is

$$\widehat{K}^2 = \frac{K^2}{F^2} + \frac{2n\,\epsilon\,\Lambda_{(n)}\,\lambda^2}{(n-2)\,F} + \frac{n(n+1)\,\lambda^4}{8}. \tag{61}$$

Thus, there are no singularities in $(n+1)$ dimensions, except for those which come from the embedded spacetime. The corresponding scalar field and it potential can be written as

$$\Psi(y) = \frac{2}{\lambda}\sqrt{\frac{2\,\epsilon\,\Lambda_{(n)}}{(n-2)\,F_0}}\left(1 - \mathrm{e}^{-\lambda y/2}\right) + \Psi_0 \tag{62}$$

$$V(\Psi) = -\frac{(n-1)\,\Lambda_{(n)}}{(n-2)\,F_0} + \frac{\epsilon\,\lambda(n-1)}{2}\left[\sqrt{\frac{2\,\epsilon\,\Lambda_{(n)}}{(n-2)\,F_0}}\,(\Psi-\Psi_0) - \frac{\lambda}{4}[n + (\Psi-\Psi_0)^2]\right]. \tag{63}$$

- In the limit $\lambda \to 0$ these expressions reduce to (49) and (50), respectively.
- If $\Lambda_{(n)} = 0$, then $\Psi = \Psi_0$ and V is constant. In this case the energy-momentum tensor (3) reduces to

$$^{(n+1)}T_{AB} = \frac{\epsilon\,n(n-1)\,\lambda^2}{8}\,\gamma_{AB}. \tag{64}$$

Thus, when $\Lambda_{(n)} = 0$ the embedding space is an Einstein space with an effective cosmological constant

$$\Lambda_{(n+1)} = \frac{\epsilon\,n(n-1)\,\lambda^2}{8}, \tag{65}$$

which implies $\lambda = \pm\sqrt{C}$, where C is given by (28).

Example II: Motivated by (33) we now choose the functional form

$$F(y) = \frac{F_0}{(1+b)^2}\, \mathrm{e}^{\lambda y} \left(1 + b\, \mathrm{e}^{-\lambda y}\right)^2, \quad b \geq 0, \tag{66}$$

which for $\lambda \to 0$ reduces to the case $F = F_0$ given by (49)-(50), and for $b \to 0$ gives back (60)-(63). In fact, substituting this into (15) and choosing the integration constant appropriately we get

$$\Psi(y) \;=\; -\frac{2A}{\lambda\sqrt{b}} \left[\arctan\left(\sqrt{b}\, \mathrm{e}^{-\lambda y/2}\right) - \arctan\left(\sqrt{b}\right)\right] + \Psi_0 \tag{67}$$

where

$$A^2 = \frac{2\,\epsilon\,\Lambda_{(n)}\,(1+b)^2}{(n-2)\,F_0} - \lambda^2\, b\,(n-1). \tag{68}$$

Clearly this equation requires $\epsilon\Lambda_{(n)} > 0$. The corresponding potential can be written as

$$V(\Psi) = -\frac{\epsilon}{8\,b}\,(n-1)\,\left[A^2\, \sin^2\alpha(\Psi) + n\lambda^2 b\right], \tag{69}$$

with

$$\alpha(\Psi) = \frac{\lambda\sqrt{b}}{A}\,(\Psi - \Psi_0) - 2\arctan(\sqrt{b}). \tag{70}$$

• For $A = 0$, the field and the potential are constant. In this case the energy-momentum tensor in $(n+1)$ dimension reduces to (64) and $\lambda = \pm\sqrt{C}$, where C is given by (28). Thus, $A = 0$ implies

$$b = \frac{\epsilon\,\bar{\Lambda}_{(n)}}{3\,C}\, \frac{(1+b)^2}{F_0}$$

After a rescaling $F_0 \to F_0\,(1+b)^2$, the function F given by (66) becomes identical to (33).

Thus, the function F does not determine, in a unique way, the energy-momentum tensor in $(n+1)$-dimensions; choosing F in such a way that it has the same functional form as in an Einstein embedding, we can generate a new family of embeddings, with nontrivial Ψ and V, which contains the original one as a limiting case. This is something one should have expected a priori, because in general relativity the same geometry can be engendered by different material distributions.[4]

Example III: As a final example we consider the embedding of vacuum solutions of n-dimensional general relativity with $\Lambda_{(n)} = 0$. In this case (15) can be formally integrated as

$$F(y) = F_0\, \mathrm{e}^{a\,y}\, \mathrm{e}^{-\int f(y)\,dy}, \tag{71}$$

where F_0 and a are constants of integration and $f(y)$ is given by

$$\Psi'^2 = \frac{(n-1)}{2}\, f'. \tag{72}$$

Substituting (71) into (14) we find

$$\epsilon\, V = \frac{(n-1)}{8}\, \left[2\,f' - n\,(a-f)^2\right]. \tag{73}$$

If we set $f = 0$ we regain the Einstein embeddings of Type II with $a = \pm\sqrt{C}$.

[4] A nice example is provided by the FRW dust models which can be interpreted as exact solutions of the Einstein field equations for a viscous fluid [50].

To complete the embedding we need to specify the function f. We wish to generalize the model given in [45]. To this end we choose

$$f(y) = -2\,p\lambda\,(1+\lambda\,y)^m, \tag{74}$$

where $(p,\,\lambda,\,m)$ are some constant parameters. In fact, substituting this into (72) we get

$$\Psi'^2 = -\frac{m\,p\,(n-1)\,\lambda^2}{(1+\lambda\,y)^{1-m}}. \tag{75}$$

When $m=-1$ this expression reduces to (21) and Ψ is given by (22). For other values of m the solution is

$$\Psi = \bar{q}\,(1+\lambda\,y)^{(m+1)/2} + \Psi_0, \quad m \neq -1, \tag{76}$$

where Ψ_0 is a constant of integration and

$$\bar{q} = \frac{2\sqrt{(-m\,p)(n-1)}}{m+1}.$$

The embedding function corresponding to (74) is given by

$$F(y) = F_0\,\mathrm{e}^{ay}\ \times \exp\left[\frac{2\,p\,(1+\lambda\,y)^{m+1}}{m+1}\right], \quad m \neq -1, \tag{77}$$

and

$$F(y) = F_0\,\mathrm{e}^{ay}\ \times (1+\lambda\,y)^{2p}, \quad m = -1. \tag{78}$$

For $m=-1$ the scalar field is identical to (22). However, the potential is not equal to (24) - unless $a=0$.

For $a=0$ and $m\neq 1$ the potential has the form

$$\epsilon\,V = -\frac{(n-1)\,p\,\lambda^2}{2}\,[n\,p\,Z^{2m} + m\,Z^{m-1}], \quad Z = \left(\frac{\Psi-\Psi_0}{\bar{q}}\right)^{2/(m+1)}, \quad m \neq -1. \tag{79}$$

The above examples demonstrate that the embeddings provided by the metric (16) are consistent with higher-dimensional scalar fields with different potentials (not only exponential), in contrast with what is claimed in [45].

6 Final remarks

We have shown that all vacuum solutions of general relativity in n-dimensions, with or without a cosmological constant $\Lambda_{(n)}$, can in several nonequivalent ways be embedded in a $(n+1)$-dimensional space which is either an Einstein space or a solution of the higher-dimensional field equations whose source is a real massless scalar field Ψ, minimally-coupled to Einstein gravity and self-interacting through a potential $V(\Psi)$. This work generalizes a number of results discussed in the literature [43–48].

From (16) it follows that all the hypersurfaces orthogonal to the unit vector $\hat{n}^A = \delta^A_y$ inherit the line element

$$dS^2\big|_\Sigma = ds^2 = F(y)\,g_{ab}\,dx^a\,dx^b. \tag{80}$$

Since ${}^{(n)}R_{\mu\nu}$ is invariant under a constant conformal transformation of the metric, the effective cosmological constant in n-dimensions, say $\Lambda^{eff}_{(n)}$, is

$$\Lambda^{eff}_{(n)} = \frac{\Lambda_{(n)}}{F(y)}. \tag{81}$$

This is a consequence of the fact that the induced metric (80) has a scaling symmetry; if we perform a coordinate transformation

$$d\bar{x}^a = \sqrt{F}\,dx^a,$$

then, in terms of the barred coordinates, $ds^2 = \bar{g}_{ab}\, d\bar{x}^a\, d\bar{x}^b$. The metric $\bar{g}_{ab} = \bar{g}_{ab}(\bar{x})$ has the same form as g_{ab} but with rescaled constants. As an illustration let us consider an example. For simplicity we examine the Schwarzschild-de Sitter metric in spherical space coordinates[5] $(r,\, \theta,\, \phi)$

$$ds^2 = hdt^2 - \frac{dr^2}{h} - r^2\,(d\theta^2 + \sin^2\theta d\phi^2), \tag{82}$$

with

$$h = 1 - \frac{2M}{r} - \frac{\Lambda_{(4)}}{3}\, r^2, \tag{83}$$

which is a solution of the four-dimensional Einstein field equations ${}^{(4)}G_{\mu\nu} = \Lambda_{(4)}\, g_{\mu\nu}$. The two parameters M and $\Lambda_{(4)}$ represent the mass of the central body and the cosmological constant, respectively.

The five-dimensional spacetime with metric

$$dS^2 = \gamma_{AB}\, dx^A dx^B = F(y)\, \left(hdt^2 - \frac{dr^2}{h} - r^2\, d\Omega^2\right) + \epsilon dy^2, \tag{84}$$

where F is any of the functions discussed in the previous sections, provides an embedding for the Schwarzschild-de Sitter metric. In fact, on any hypersurface Σ orthogonal to y, after the transformation of coordinates

$$\bar{r} = \sqrt{F}\, r, \qquad \bar{t} = \sqrt{F}\, t$$

the metric (84) reduces to

$$dS^2\big|_{\Sigma} = ds^2 = \left(1 - \frac{2\,M^{eff}}{\bar{r}} - \frac{\Lambda^{eff}_{(4)}}{3}\,\bar{r}^2\right)\, d\bar{t}^2 - \left(1 - \frac{2\,M^{eff}}{\bar{r}} - \frac{\Lambda^{eff}_{(4)}}{3}\,\bar{r}^2\right)^{-1}\, d\bar{r}^2 - \bar{r}^2\, d\Omega^2, \tag{85}$$

where the effective mass M^{eff} and cosmological constant $\Lambda^{eff}_{(4)}$ in 4D depend on the extra coordinate, viz.,

$$M^{eff} = M\sqrt{F(y)}, \qquad \Lambda^{eff}_{(n)} = \frac{\Lambda_{(4)}}{F(y)}. \tag{86}$$

If physics takes place on a hypersurface of constant y, then M and $\Lambda_{(4)}$ are constants. However, if we adopt the approach used elsewhere, in which $y(s)$ is given by a solution of the 5D geodesic equation, then these quantities are no longer constants but vary with time [42–44, 52]. This approach can bring to light important physical effects, e.g., in the early universe and galaxy formation. The study of such effects is beyond the scope of this work.

Finally, we note that a nonvanishing logarithmic derivative of the embedding function F gives rise to an extra (non-gravitational) force acting in n-dimensions. This follows from the assumption that massless particles in $(n+1)$ move along null geodesics, in the same way as photons in 4D. Indeed, the geodesic equation in the embedding higher-dimensional space is

$$\frac{d^2x^A}{d\chi^2} + K^A_{BC}\frac{dx^B}{d\chi}\frac{dx^C}{d\chi} = 0, \tag{87}$$

where χ is some affine parameter along the geodesic. To obtain the n-dimensional part of the geodesic equation (87) we set $A = a = 0,\, 1, ...,\, n$ and introduce the function l as

$$d\chi = lds. \tag{88}$$

After some manipulations we get

$$\frac{d^2x^a}{ds^2} + \Gamma^a_{bc}\,\frac{dx^b}{ds}\frac{dx^c}{ds} = \left[\frac{1}{l}\,\frac{dl}{ds} - \frac{F'}{F}\,\frac{dy}{ds}\right]\,\frac{dx^a}{ds}. \tag{89}$$

[5]This applies to more general solutions in vacuum, e.g., Kerr or Kerr-de Sitter in any number of dimensions [51]

Now setting $A = y$, and using (80), we find

$$\frac{d^2y}{ds^2} - \frac{1}{l}\left(\frac{dl}{ds}\right)\left(\frac{dy}{ds}\right) - \frac{\epsilon\, F'}{2\, F} = 0. \tag{90}$$

If $\epsilon = -1$, then the assumption $dS^2 = 0$ implies $ds^2 = dy^2 > 0$ which corresponds to a timelike geodesic of a massive particle in n-dimensions. Along such geodesics we can take $(dy/ds) = 1$. Finally, combining (89)-(90) we obtain

$$\frac{d^2x^a}{ds^2} + \Gamma^a_{bc}\frac{dx^b}{ds}\frac{dx^c}{ds} = -\frac{F'}{2F}\frac{dx^a}{ds}. \tag{91}$$

The term on the right-hand side modifies the conventional geodesic motion. It represents an extra force per unit mass (a fifth force) that causes an anomalous acceleration, which in principle could be detected experimentally.

Appendix A: Evaluation of the Kretschmann scalar

To investigate the presence of singularities in the embedding spacetime we study the Kretschmann scalar

$$\widehat{K}^2 = \widehat{R}_{ABCD}\,\widehat{R}^{ABCD}, \tag{A-1}$$

where $\widehat{R}_{ABCD} = {}^{(n+1)}R_{ABCD}$ is the Riemann tensor in $(n+1)$ - dimensions. For the embeddings under consideration, generated by the line element (16), Christoffel symbols (6) reduce to

$$\begin{aligned} K^y_{yy} &= K^a_{yy} = K^y_{ay} = 0 \\ K^b_{ya} &= \frac{F'}{2F}\,\delta^b_a, \quad K^y_{ab} = -\frac{\epsilon}{2}\,F'\,g_{ab}, \quad K^c_{ab} = \Gamma^c_{ab}. \end{aligned} \tag{A-2}$$

As a consequence the Riemann tensor in $(n+1)$-dimensions can be decomposed as

$$\widehat{R}_{abcd} = F\,R_{abcd} + \epsilon\,\frac{F'^2}{4}\,(g_{bc}\,g_{ad} - g_{bd}\,g_{ac}) \tag{A-3}$$

$$\widehat{R}_{abcy} = 0 \tag{A-4}$$

$$\widehat{R}_{ayby} = -\frac{1}{2}\,g_{ab}\left(F'' - \frac{F'^2}{2F}\right). \tag{A-5}$$

Here $R_{abcd} = {}^{(n)}R_{abcd}$ is the Riemann curvature tensor constructed with the metric g_{ab}. To evaluate (A-1) we note that $\widehat{K}^2 = \left(\widehat{R}_{abcd}\,\widehat{R}^{abcd} + 4\,\widehat{R}_{ayby}\,\widehat{R}^{ayby}\right)$, with

$$\widehat{R}_{abcd}\,\widehat{R}^{abcd} = \frac{1}{F^2}\,R_{abcd}\,R^{abcd} - \frac{\epsilon\,F'^2}{F^3}\,R + \frac{n(n-1)}{8}\left(\frac{F'}{F}\right)^4 \tag{A-6}$$

$$\widehat{R}_{ayby}\,\widehat{R}^{ayby} = \frac{n}{4F^2}\left(F'' - \frac{F'^2}{2F}\right)^2, \tag{A-7}$$

where R is the scalar curvature of the n-dimensional space, i.e., $R = g^{ac}\,g^{bd}\,R_{abcd}$. For the spacetime described by (1) it is $R = -[2\,n\,\Lambda_{(n)}/(n-2)]$. Thus, using the above expressions we obtain

$$\widehat{K}^2 = \frac{K^2}{F^2} + n\left[\left(\frac{F''}{F}\right)^2 - \frac{F''\,F'^2}{F^3} + \frac{(n+1)}{8}\left(\frac{F'}{F}\right)^4 + \frac{2\,\epsilon\,\Lambda_{(n)}}{n-2}\,\frac{F'^2}{F^3}\right], \tag{A-8}$$

with $K^2 = R_{abcd}\,R^{abcd}$.

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